Generating kernel for the boson realisation of symplectic algebras

1985 J. Phys. A: Math. Gen. 18 L493

(http://iopscience.iop.org/0305-4470/18/9/001)

J. Phys. A: Math. Gen. **18** (1985) L493–L498. Printed in Great Britain

LETTER TO THE EDITOR

Generating kernel for the boson realisation of symplectic algebras

O Castaños†, P Kramer‡ and M Moshinsky§‖

† Centro de Estudios Nucleares, UNAM Apdo. Postal 70-543, México, DF, 04510 México
‡ Institut für Theoretische Physik Universität Tübingen, West Germany
§ Instituto de Física, UNAM Apdo. Postal 20-364, México, DF, 01000 México

Received 9 April 1985

Abstract. The discussion of boson realisations of symplectic algebras requires as an essential ingredient an operator K needed for the passage from the Dyson to the Holstein–Primakoff realisation. In previous papers the matrix form of K^2 was derived through appropriate recursion relations. In the present analysis we show that K^2 can be determined in explicit and closed form as the overlap of coherent states of the symplectic group. The matrix elements of K^2 can then be obtained by expanding this overlap in terms of appropriate eigenstates.

In the last few years there has been considerable interest in the symplectic algebras and their boson realisations (Rowe *et al* 1984, Rowe 1984, Deenen and Quesne 1982, 1983, 1984, 1985, Kramer 1982a, b, Kramer *et al* 1983, 1984, Moshinsky 1984, 1985, Moshinsky *et al* 1984, Castaños *et al* 1985) particularly because of their application to the symplectic model of the nucleus (Rosensteel and Rowe 1980). In this letter we show that the operator K^2, to be defined below, which is an essential ingredient of the boson realisation (Rowe *et al* 1984, Deenen and Quesne 1982, 1985, Moshinsky *et al* 1984), can be expressed as a kernel with respect to appropriate coherent states, and that a closed analytic expression for this kernel can be obtained from considerations of finite symplectic transformations. By developing this kernel in terms of appropriate wavefunctions that are polynomials in the complex variables appearing in the coherent states, one can then get the matrix representation of K^2 without going through the procedure of solving recursion relations (Rowe *et al* 1984, Moshinsky *et al* 1984).

For the sake of simplicity we shall restrict our discussion to sp(4, R) which already encompasses the general features of the problem, as it includes 'open shell' cases (Castaños *et al* 1985). The ten generators of sp(4, R) are then given (Goshen and Lipkin 1968, Castaños *et al* 1985) by

$$\mathfrak{N},\ B_i^+,\ J_i,\ B_i; \qquad i=1,2,3 \tag{1}$$

satisfying the commutation rules

$$[B_i^+, B_j^+]=0, \qquad [B_i, B_j]=0 \tag{2a, b}$$

$$[B_i, B_j^+]=-2\mathrm{i}\varepsilon_{ijk}J_k+2\delta_{ij}\mathfrak{N} \tag{2c}$$

‖ Member of El Colegio Nacional and Instituto Nacional de Investigaciones Nucleares.

0305-4470/85/090493+06\$02.25

$$[J_i, B_j^+] = i\varepsilon_{ijk}B_k^+, \qquad [J_i, B_j] = i\varepsilon_{ijk}B_k, \qquad [J_i, J_j] = i\varepsilon_{ijk}J_k \qquad (2d, e, f)$$

$$[\mathfrak{N}, B_i^+] = B_i^+, \qquad [\mathfrak{N}, B_i] = -B_i, \qquad [\mathfrak{N}, J_i] = 0 \qquad (2g, h, i)$$

where ε_{ijk} is the antisymmetric symbol and repeated indices are summed from 1 to 3.

The set (1) of ten generators can be divided into three subsets of raising, weight and lowering type which are separated below by semicolons, i.e.

$$B_i^+, J_+; \mathfrak{N}, J_0; B_i, J_-, \qquad i = 1, 2, 3 \qquad (3)$$

where

$$J_\pm = J_1 \pm iJ_2, \qquad J_0 = J_3. \qquad (4)$$

The lowest weight state which we designate by $|ws\rangle$ is then given by the solution of the equations

$$B_i|ws\rangle = 0 \qquad i = 1, 2, 3, \qquad J_-|ws\rangle = 0 \qquad (5a, b)$$

$$\mathfrak{N}|ws\rangle = w|ws\rangle, \qquad J_0|ws\rangle = -s|ws\rangle \qquad (5c, d)$$

where w, s characterise the irreducible representation (irrep) of sp(4, R) which we are dealing with. A complete but non-orthonormalised set of states corresponding to this irrep (w, s) may be written as

$$|mn\rangle \equiv (B_1^+)^{n_1}(B_2^+)^{n_2}(B_3^+)^{n_3}J_+^m|ws\rangle. \qquad (6)$$

Applying then the generators of sp(4, R) to the states $|mn\rangle$ and using the commutation rules (2) (Castaños *et al* 1985) we showed that these generators can be expressed symbolically in terms of multiplicative operators B_i^+, differential ones $\partial/\partial B_i^+$ and operators J_i acting only on the $J_+^m|ws\rangle$ part of the states. From this analysis a Dyson type realisation of the generators of sp(4, R) was given which, in vector notation, takes the form

$$\boldsymbol{B}^+ = \boldsymbol{\beta}^+, \qquad \boldsymbol{J} = \boldsymbol{l} + \boldsymbol{s}, \qquad \mathfrak{N} = \mathcal{N} + w \qquad (7a, b, c)$$

$$\boldsymbol{B} = -\boldsymbol{\beta}^+(\boldsymbol{\beta}\cdot\boldsymbol{\beta}) + (2\mathcal{N} + 2w)\boldsymbol{\beta} - 2i(\boldsymbol{\beta}\times\boldsymbol{s}) \qquad (7d)$$

where

$$\boldsymbol{l} = -i(\boldsymbol{\beta}^+\times\boldsymbol{\beta}), \qquad \mathcal{N} = \boldsymbol{\beta}^+\cdot\boldsymbol{\beta}, \qquad (8a, b)$$

and in which the components of the vectors $\boldsymbol{\beta}^+$, $\boldsymbol{\beta}$, $\boldsymbol{s}$ satisfy the commutation relations

$$[\beta_i, \beta_j] = [\beta_i^+, \beta_j^+] = [\beta_i^+, s_j] = [\beta_i, s_j] = 0 \qquad (9a)$$

$$[\beta_i, \beta_j^+] = \delta_{ij}, \qquad [s_i, s_j] = i\varepsilon_{ijk}s_k. \qquad (9b, c)$$

It is then easy to check that the generators $\mathfrak{N}$, B_i^+, B_i, J_i of sp(4, R) given by (7) satisfy the commutation relations (2) if β_i^+, β_i, s_i satisfy (9). Note that (9) would also be the commutation relations for the generators of the direct sum $\omega(3)\oplus$su(2) of a three-dimensional Weyl Lie algebra $\omega(3)$ and an independent su(2) algebra.

The β_i^+, β_i can not be interpreted as bosons because, from their appearance in (7) where $(B_i^+)^+ = B_i$, they are not Hermitian conjugate i.e. $(\beta_i^+)^+ \neq \beta_i$. Thus from the Dyson realisation (7) in terms of the β_i^+, β_i, s_i we have to pass to a Holstein–Primakoff one in terms of the operators

$$b_i^+, \qquad b_i, \qquad S_i \qquad (10)$$

satisfying the same commutation rules (9) but now having in addition the Hermiticity conditions

$$(b_i^+)^\dagger = b_i, \qquad S_i^\dagger = S_i. \tag{11a, b}$$

These new operators will then be related with the previous ones by a similarity transformation (Rowe *et al* 1984, Deenen and Quesne 1985, Moshinsky *et al* 1984, Castaños *et al* 1985)

$$b_i^+ = K^{-1}\beta_i^+ K, \qquad b_i = K^{-1}\beta_i K, \qquad S_i = K^{-1}s_i K \tag{12a, b, c}$$

which will preserve the commutation rules (9) and where K will be determined through appropriate Hermiticity conditions.

As $(\beta_i^+)^\dagger \neq \beta_i$ while $(b_i^+)^\dagger = b_i$ the K is not unitary but we can assume, without loss of generality, that it is Hermitian and invariant under the u(2) subalgebra of sp(4) i.e.

$$K^\dagger = K, \qquad [K, \mathfrak{N}] = [K, J_i] = 0. \tag{13}$$

From equations (7) and (12) we then obtain

$$\boldsymbol{B}^+ = K\boldsymbol{b}^+K^{-1}, \qquad \boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{14a, b}$$

$$\boldsymbol{B} = K[-\boldsymbol{b}^+(\boldsymbol{b}\cdot\boldsymbol{b}) + (2N + 2w)\boldsymbol{b} - 2\mathrm{i}(\boldsymbol{b}\times\boldsymbol{S})]K^{-1} \tag{14c}$$

$$\mathfrak{N} = N + w, \tag{14d}$$

in which

$$\boldsymbol{L} = -\mathrm{i}(\boldsymbol{b}^+\times\boldsymbol{b}), \qquad N = \boldsymbol{b}^+\cdot\boldsymbol{b}, \tag{15a, b}$$

and where to obtain (14*b*, *d*) we passed K, K^{-1} to the left-hand side and made use of (13).

As $\boldsymbol{B}^+$ is the Hermitian conjugate of $\boldsymbol{B}$ and K is Hermitian we obtain

$$\boldsymbol{B} = K^{-1}\boldsymbol{b}K \tag{16}$$

which combined with (14*c*) gives us the equation

$$\boldsymbol{b}K^2 = K^2[-\boldsymbol{b}^+(\boldsymbol{b}\cdot\boldsymbol{b}) + (2N+2w)\boldsymbol{b} - 2\mathrm{i}(\boldsymbol{b}\times\boldsymbol{S})]. \tag{17a}$$

The equation (17*a*) together with

$$NK^2 - K^2N = 0, \qquad (L_i + S_i)K^2 - K^2(L_i + S_i) = 0 \tag{17b, c}$$

which follow from (13), (14*b*, *d*), are then the ones that determine K^2.

Thus far the analysis is the standard one (Rowe *et al* 1984, Deenen and Quesne 1985, Moshinsky *et al* 1984, Castaños *et al* 1985). What is new is that we will now consider first the coherent states of the $\omega(3)\oplus\mathrm{su}(2)$ Lie algebra defined by

$$|y, z) \equiv \exp(\bar{\boldsymbol{z}}\cdot\boldsymbol{b}^+)\exp(\bar{y}S_+)|s) \tag{18}$$

where y, z_i, $i = 1, 2, 3$ are complex numbers, $\bar{y}$, $\bar{z}_i$ their conjugates, and the ket$|s)$ is the direct product

$$|s) = |0\} \times |s - s\}, \tag{19}$$

where $|0\}$ is the 0 quantum state associated with the bosons b_i^+, b_i while $|s-s\}$ is the lowest weight state of the independent spin operator S_i.

We now take the operator equations (17) between the ket$|y, z)$ and a bra

$$(y'z'| = (s|\exp(y'S_-)\exp(z'\cdot \boldsymbol{b}). \tag{20}$$

Insofar as the behaviour of the operators with respect to the variables z_i we are in standard Bargmann-Hilbert space (Bargmann 1962, 1971) in which the operators b_i^+, b_i are replaced by z_i, $\partial/\partial z_i$, while when dealing with y we are in the coherent states representation of su(2) (Kramer and Saraceno 1981) where the differential operators associated with the S_i are given in the last reference. Thus from (17) we can obtain straightforwardly that the kernel

$$(y'z'|K^2|yz) \tag{21}$$

satisfies the differential equations

$$(\partial/\partial z_i')(y'z'|K^2|yz) = [-(\bar{z}\cdot\bar{z})\partial/\partial\bar{z}_i + \bar{z}_i(2\bar{N}+2w) + 2\mathrm{i}(\bar{z}\times\bar{\boldsymbol{S}})_i](y'z'|K^2|yz), \tag{22a}$$

$$(N'-\bar{N})(y'z'|K^2|yz) = 0, \tag{22b}$$

$$(L_i'+S_i'+\bar{L}_i+\bar{S}_i)(y'z'|K^2|yz) = 0 \tag{22c}$$

where

$$N' = z_j'\partial/\partial z_j', \qquad \bar{N} = \bar{z}_j\partial/\partial\bar{z}_j, \tag{23a, b}$$

$$L_i' = -\mathrm{i}\varepsilon_{ijk}z_j'\partial/\partial z_k', \qquad \bar{L}_i = -\mathrm{i}\varepsilon_{ijk}\bar{z}_j\partial/\partial\bar{z}_k, \tag{23c, d}$$

with repeated indices summed from 1 to 3. The operators S_i', $\bar{S}_i$, $i = 1, 2, 3$ are given first through the spherical components

$$S_1 = \tfrac{1}{2}(S_+ + S_-), \qquad S_2 = -\tfrac{1}{2}\mathrm{i}(S_+ - S_-), \qquad S_3 = S_0 \tag{24}$$

where (Kramer and Saraceno 1981) we now have

$$S_+' = y'(2s - y'\partial/\partial y'), \qquad S_0' = (y'\partial/\partial y' - s), \qquad S_-' = \partial/\partial y' \tag{25}$$

$$\bar{S}_+ = -\partial/\partial\bar{y}, \qquad \bar{S}_0 = (s - \bar{y}\partial/\partial\bar{y}), \qquad \bar{S}_- = -\bar{y}(2s - \bar{y}\partial/\partial\bar{y}). \tag{26}$$

At first sight the set of differential equations (22) seems very difficult to solve, but we shall proceed to find the solution with the help of the coherent states of the sp(4) Lie algebra defined by

$$|y, z\rangle = \exp(\bar{z}\cdot\boldsymbol{B}^+)\exp(yJ_+)|ws\rangle \tag{27}$$

where we note their angular bracket form to distinguish them from the round bracket ones $|yz)$ associated with the $\omega(3)\oplus$su(2) Lie algebra.

We now consider the matrix element

$$\langle y'z'|B_i|yz\rangle = (\partial/\partial z_i')\langle y'z'|yz\rangle \tag{28a}$$

where the right-hand side follows immediately as the overlap of two coherent states of sp(4) is given by

$$\langle y'z'|yz\rangle = \langle ws|\exp(y'J_-)\exp(z'\cdot\boldsymbol{B})\exp(\bar{z}\cdot\boldsymbol{B}^+)\exp(\bar{y}J_+)|ws\rangle. \tag{29}$$

The left-hand side of (28a) can be written, by the use of the commutators (2) and the properties (5) of $|ws\rangle$, as a differential equation in $\bar{y}$, $\bar{z}$ of exactly the same form as the right-hand side of (22a). In a similar fashion we get from the matrix elements

$$\langle y'z'|\mathfrak{N}|yz\rangle, \qquad \langle y'z'|J_i|yz\rangle \tag{28b, c}$$

that the overlap $\langle y'z'|yz\rangle$ satisfies also the equations (22*b*), (22c). Thus we have

$$(y'z'|K^2|yz) = \langle y'z'|yz\rangle. \tag{30}$$

It now remains to evaluate the overlap $\langle y'z'|yz\rangle$ of (29), but we note that it is the matrix element with respect to the lowest weight state $|ws\rangle$ of exponentials of generators of sp(4) and thus it is related with representations of finite symplectic transformations that have been evaluated by Kramer (1985) and by Quesne (1985). Particularising their results to the case of sp(4, R) in the vector notation used here for the variables, one obtains

$$\langle y'z'|yz\rangle = M^{2s}[1-2(z'\cdot\bar{z})+(z'\cdot z')(\bar{z}\cdot\bar{z})]^{-w-s} \tag{31a}$$

$$M = (1+y'\bar{y})(1-z'\cdot\bar{z})+\mathrm{i}\sqrt{2}\boldsymbol{v}\cdot(z'\times\bar{z}) \tag{31b}$$

and the vector $\boldsymbol{v}$ has components

$$v_1 = -\tfrac{1}{2}\sqrt{2}(y'+\bar{y}), \qquad v_2 = \tfrac{1}{2}\mathrm{i}\sqrt{2}(y'-\bar{y}), \qquad v_3 = \tfrac{1}{2}\sqrt{2}(1-y'\bar{y}). \tag{31c}$$

The validity of this result can be verified by the explicit application of the differential operators appearing in the equations (22).

Thus we have an explicit analytic expression for K^2 as a kernel $(y'z'|K^2|yz)$ with respect to coherent states of the $\omega(3)\oplus\mathrm{su}(2)$ Lie algebra. To get the K^2 in matrix form we note that the orthonormal states for this group can be written as

$$(yz|nlsjm) = \sum_{\mu\sigma} \langle l\mu, s\sigma|jm\rangle A_{nl}(z\cdot z)^{(n-1)/2}\mathcal{Y}_{lm}(z)\left(\frac{(2s!)}{(s+\sigma)!(s-\sigma)!}\right)^{1/2} y^{s+\sigma} \tag{32}$$

where $\mathcal{Y}_{lm}(z)$ is a solid harmonic, the $\langle l\rangle$ is a Clebsch-Gordan coefficient and the normalisation (Moshinsky 1969) A_{nl} is given by

$$A_{nl} = (-1)^{(n-1)/2}(4\pi)^{1/2}[(n+l+1)!!(n-l)!!]^{-1/2}. \tag{33}$$

Note that the part of the state associated with y, i.e., for the $\boldsymbol{S}$, is normalised according to the prescription given by Kramer and Saraceno (1981) for the coherent state analysis of su(2).

Clearly then if we expand

$$y'z'|K^2|yz) = \sum_{njm}\sum_{ll'}\{(y'z'|nl'sjm)(nl'sjm|K^2|nlsjm)(nlsjm|yz)\} \tag{34}$$

the coefficients $(nl'sjm|K^2|nlsjm)$ are the matrix elements we need to determine (Castaños *et al* 1985).

We plan to give a more complete version of this analysis, including the explicit determination of $(nl'sjm|K^2|nlsjm)$, in a future paper. It is also clear that the extension to sp($2d$, R), where d is an arbitrary integer, is, at least conceptually, straightforward, and a series of papers are planned in relation with sp(6, R).

References

Bargmann V 1962 *Rev. Mod. Phys.* **34** 529
—— 1971 *Commun. Pure Appl. Math.* **21** 41
Castaños O, Chacón E, Moshinsky M and Quesne C 1985 *J. Math. Phys.* **26** to be published
Deenen J and Quesne C 1982 *J. Math. Phys.* **23** 878, 2004
—— 1983 *J. Phys. A: Math. Gen.* **16** 2095

—— 1984 *J. Math. Phys.* **25** 1638, 2354
—— 1985 *Boson Representation of the Real Symplectic Group and Their Application to the Nuclear Collective Model, Université Libre de Bruxelles preprint PTM 84-07*
Goshen S and Lipkin H J 1968 in *Spectroscopic and Group Theoretical Methods in Physics, Racah Memorial Volume* (Amsterdam: North-Holland)
Kramer P and Saraceno M 1981 *Geometry of the Time-Dependent Variational Principle in Quantum Mechanics* (Berlin: Springer)
Kramer P 1982a *Ann. Phys., NY* **141** 254
—— 1982b *Ann. Phys., NY* **141** 259
—— 1985 *Hilbert spaces of analytic functions and representations of the positive discrete series of Sp(6, R)* to be published
Kramer P, Papadopoulos Z and Schweizer W 1983 *Ann. Phys., NY* **149** 44
—— 1984 *Nucl. Phys.* A **431** 75
—— 1985 *Irreducible representations of the symplectic group in Hilbert spaces of analytic functions and associated Dyson and Holstein Primakoff realizations* to be published
Moshinsky M 1969 *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks* (New York: Gordon and Breach) p 22
—— 1984 *Nucl. Phys.* A **421** 81
—— 1985 *J. Phys. A: Math. Gen.* **18** L1
Moshinsky M, Chacón E and Castaños O 1984 in *Proc. XIII Colloq. on Group Theoretical Methods in Physics* (Singapore: World Scientific)
Quesne C 1985 *Coherent states of the real symplectic group in a complex analytic parametrization I. Perelomov coherent states* to be published
Rosensteel G and Rowe D J 1980 *Ann. Phys., NY* **126** 198, 343
Rowe D J 1984 *J. Math. Phys.* **25** 2662
Rowe D J, Rosensteel G and Carr R 1984 *J. Phys. A: Math Gen.* **17** L399